CHAPTER 8

NUMERICAL INTEGRATION

Integration plays an important role in many fields of science and engineering. For applications, numerical values of integrals are often required. However, in many cases, the evaluation of integrals, or quadrature, by elementary functions may not be feasible. Hence, approximating the value of an integral in a reliable fashion is a problem of utmost importance.

Numerical quadrature is in fact one of the oldest branches of mathematics: the determination, approximately or exactly, of the areas of regions bounded by lines or curves, a subject which was studied by the ancient Babylonians. The word ‘‘quadrature’’ indicates the process of measuring an area inside a curve by finding a square having the same area. Probably no other problem has exercised a greater or a longer attraction than that of constructing a square equal in area to a given circle. Thousands of people have worked on this problem, including the ancient Egyptians as far back as 1800B.C.

In this chapter, we provide an exposition of methods for approximating integrals, including those that are multidimensional.

1. The Trapezoidal Method►

Here, we will discuss the trapezoidal rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

If the interval in question is partitioned in to equal width subintervals. That is if you let $[a, b]$ be partitioned by

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

then the composite trapezoidal rule is

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx \cong \frac{1}{2}\sum_{i=0}^{n-1} (x_{i+1} - x_i)[f(x_{i+1}) + f(x_i)]$$

Since each subinterval has equal width $x_{i+1} - x_i = h$, and we have

$$\int_a^b (x)dx \cong \frac{h}{2}\sum_{i=0}^{n-1} f(x_{i+1}) + f(x_i)$$

In your calculus class, you saw this in the less comprehensible form:

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right] - \frac{h^2(b-\mathrm{a})}{12} f''(\zeta) \qquad (1)$$

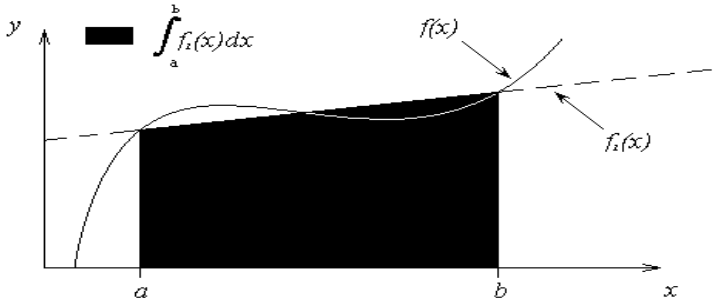


Figure1 Geometric representation of trapezoidal rule.

Example 1 ►

Use the trapezoidal rule to approximate $\int_0^2 e^{x^2} dx\,, \quad h = 0.25$

Solution

n	x_n	$f(x_n) = e^{x_n{}^2}$	w_n	$w_n f(x_n)$
0	0	1	1	1
1	0.25	1.0645	2	2.129
2	0.50	1.2840	2	2.568
3	0.75	1.7551	2	3.5102
4	1.00	2.7183	2	5.4366
5	1.25	4.7707	2	9.5414
6	1.50	9.4877	2	18.9754
7	1.75	21.3809	2	42.7618
8	2.00	54.5982	1	54.5982

$\sum$	140.5206

$$\int_0^2 e^{x^2}dx = \frac{h}{2}\sum_{n=0}^{8} w_n f(x_n) = \frac{0.25}{2}(140.5206) = 17.565075$$

The error= $\frac{h^2(b-a)}{12}f''(\zeta) = \frac{(0.25)^2(2-0)}{12}f''(\zeta) = 0.0104167f''(\zeta)$

Example 2 ►

Use the trapezoidal rule to approximate $\int_1^2 \frac{1}{x}dx$, $n = 10$

Solution

Sincs $n = 10 \rightarrow h = 0.1$

n	x_n	$f(x_n) = \frac{1}{x_n}$	w_n	$w_n f(x_n)$
0	1.0	1	1	1
1	1.1	0.9090909	2	1.8181818
2	1.2	0.8333333	2	1.6666666
3	1.3	0.7692307	2	1.5384615
4	1.4	0.7142857	2	1.4285714
5	1.5	0.6666667	2	1.3333334
6	1.6	0.6250000	2	1.2500000
7	1.7	0.5882352	2	1.1764705
8	1.8	0.5555556	2	1.1111112
9	1.9	0.5263157	2	1.0526315
10	2.0	0.5000000	1	0.5000000
$\sum$				13.875428

$$\int_1^2 \frac{1}{x}dx = \frac{h}{2}\sum_{n=0}^{10} w_n f(x_n) = \frac{0.1}{2}(13.875428) = 0.693771403$$

The error= $\frac{h^2(b-a)}{12}f''(\zeta) = \frac{(0.1)^2(2-1)}{12}f''(\zeta) = \frac{1}{1200}f''(\zeta)$

2. The Simpson's $^1/_3$ Rule►

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an extension of

Trapezoidal rule where the integrand is approximated by a second order polynomial.
One could use the Lagrange polynomial to derive Simpson's formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x)=\frac{\left(x-\frac{a+b}{2}\right)(x-b)}{\left(a-\frac{a+b}{2}\right)(a-b)}f(a)+\frac{(x-a)(x-b)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}f\left(\frac{a+b}{2}\right)+\frac{(x-a)\left(x-\frac{a+b}{2}\right)}{(b-a)\left(b-\frac{a+b}{2}\right)}f(b)$$

Integrating this function gets

$$\int_a^b f_2(x)dx=\left[\frac{\frac{x^3}{3}-\frac{(a+3b)x^2}{4}+\frac{b(a+b)x}{2}}{\left(a-\frac{a+b}{2}\right)(a-b)}f(a)+\frac{\frac{x^3}{3}-\frac{(a+b)x^2}{2}+abx}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}f\left(\frac{a+b}{2}\right)+\frac{\frac{x^3}{3}-\frac{(3a+b)x^2}{4}+\frac{a(a+b)x}{2}}{(b-a)\left(b-\frac{a+b}{2}\right)}f(b)\right]_a^b$$

$$=\frac{\frac{b^3-a^3}{3}-\frac{(a+3b)(b^2-a^2)}{4}+\frac{b(a+b)(b-a)}{2}}{\left(a-\frac{a+b}{2}\right)(a-b)}f(a)$$

$$+\frac{\frac{b^3-a^3}{3}-\frac{(a+b)(b^2-a^2)}{2}+ab(b-a)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}f\left(\frac{a+b}{2}\right)$$

$$+\frac{\frac{b^3-a^3}{3}-\frac{(3a+b)(b^2-a^2)}{4}+\frac{a(a+b)(b-a)}{2}}{(b-a)\left(b-\frac{a+b}{2}\right)}f(b)$$

Believe it or not, simplifying and factoring this large expression yields

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$=\frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

$$\int_a^b f(x)dx = \frac{h}{3}\{f_0+4f_1+2f_2+4f_3+\cdots+4f_{n-1}+f_n\} + \frac{h^4(b-a)f^{(4)}(\zeta)}{180}$$

<u>Remark:</u> (n) must be even number, $h = \frac{b-a}{n}$

Example 3►

Use composite Simpson's rule,

$$\int_a^b f(x) \approx \frac{h}{3}\big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{n-1})+f(x_n)\big)$$

$$\text{and } E \le \frac{(b-a)h^4}{180}\left|f^{(4)}(\zeta)\right| \quad \text{where} \quad a \le \zeta \le b,$$

to find the approximate value of the integration $\int_1^2 xe^x dx$ use $n = 4$. Find the upper bound of the truncation error.

Solution

We find the step $h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$.

Divide the interval $1 \le x \le 2$ by the step $h = 0.25$. So, we have the following scheme:

$x_0 = 1$	$x_1 = 1.25$	$x_2 = 1.5$	$x_3 = 1.75$	$x_4 = 2$

We put the data in vertical form as follows:

	i	x_i	$f(x) = xe^x$	w_n	$w_n\, f(x)$
	0	1	2.71828	1	2.71828
	1	1.25	4.362929	4	17.45171479
	2	1.5	6.722534	2	13.44506721
	3	1.75	10.070555	4	40.28221873
	4	2	14.7781122	1	14.7781122
$\sum$	-----	-------	----------	----------	88.67539293

$$\int_1^2 xe^x dx = \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$$

$$= \frac{0.25}{3} \times 88.67539293 = 7.389616.$$

The upper bound of the truncation error:

Since $f(x) = xe^x$, then the fourth derivative of it, w.r.t., x is given by $f^{(4)}(x) = (x+4)e^x$. Hence,

$E \leq \frac{(b-a)h^4}{180}\left|f^{(4)}(\zeta)\right| = \frac{(2-1)(0.25)^4}{180}(\zeta+4)e^{\zeta}$ where $1 \leq \zeta \leq 2$.
Clearly, the maximum value of the error occurs with $\zeta = 2$. Therefore, $E \leq 0.0000962166$.

Example 4►
Use composite Simpson's rule to find the approximate value of the integration
$\int_0^1 \frac{1}{x+1} dx$. Use $n = 4$. Approximate each number within 5-decimal places. Find the upper bound of the truncation error.

Solution

We find the step $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$.

Divide the interval $0 \le x \le 1$ by the step $h = 0.25$. So, we have the following scheme:

$x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$x_4 = 1$

We put the data in vertical form as follows:

	i	x_i	$f(x_i) = \frac{1}{x+1}$	w_n	$w_n\, f(x_i)$
	0	0	1	*1*	*1*
	1	0.25	*0.8*	*4*	*3.2*
	2	0.5	*0.66667*	*2*	*1.33333*
	3	0.75	*0.57143*	*4*	*2.28571*
	4	1	*0.5*	*1*	*0.5*
$\sum$	-----	-------	----------	----------	*8.31904*

$$\int_0^1 \frac{1}{x+1}\,dx = \frac{h}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\big)$$

$$= \frac{0.25}{3} \times 8.31904 = 0.6935.$$

The actual error:

Since the exact value of the integration $\int_0^1 \frac{1}{x+1}\,dx = \ln(x+1)|_0^1 = \ln(2) - \ln(1) = 0.69315,$

Then the actual error is given by 0.00035.

The upper bound of the truncation error:

Since $f(x) = \frac{1}{x+1}$, then the fourth derivative of it, w.r.t., x is given by $f^{(4)}(x) = \frac{24}{(x+1)^5}$. Hence,

$E \leq \frac{(b-a)h^4}{180}\left|f^{(4)}(\zeta)\right| = \frac{(1-0)(0.25)^4}{180}\frac{24}{(\zeta+1)^5}$ where $0 \leq \zeta \leq 1$.

Clearly, the maximum value of the error occurs with $\zeta = 0$. Therefore, $E \leq 0.00052$.

Example 5►

Use composite Simpson's rule to determine n and h to approximate $\int_0^{\pi} \sin x dx$ with an absolute error at most 0.00002.

Solution

Use the error formula in Simpson's rule

$$\frac{(b-a)h^4}{180}\left|f^{(4)}(\zeta)\right| = \left|\frac{\pi h^4}{180} sin\zeta\right| \leq \frac{\pi h^4}{180}(1) = \frac{\pi^5}{180\, n^4} \leq 0.00002$$

Where $max|\sin\zeta| = 1$, $h = \frac{b-a}{n} = \frac{\pi}{n}$

$$\therefore n \geq 18, \quad if\ n = 18 \quad then\ h = \frac{\pi}{18}.$$

3. The Simpson's $^3/_8$ Rule►

Simpsons 1/3 rule for integration was derived by approximating the integrand $f(x)$ with a 2nd order (quadratic) polynomial function.

$f_2(x)$

$$f_2(x) = a_0 + a_1 x + a_2 x^2 \qquad (1)$$

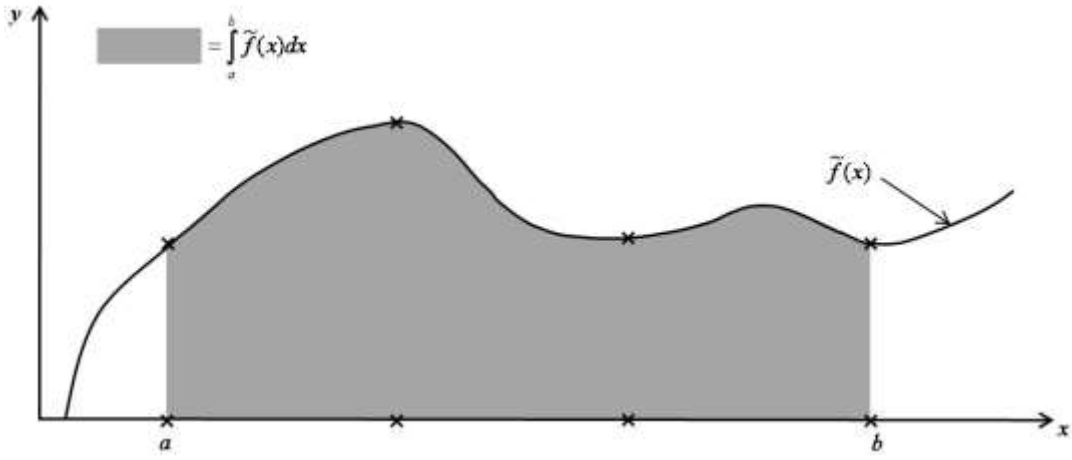


Figure 1 $\tilde{f}(x)$ Cubic function.

In a similar fashion, Simpson 3/8 rule for integration can be derived by approximating the given function $f(x)$ with the 3rd order (cubic) polynomial $f_3(x)$

$$\left.\begin{aligned} f_3(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ &= \{1, x, x^2, x^3\} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{aligned}\right\} \qquad (2)$$

which can also be symbolically represented in Figure 1.

The Simpson 3/8 rule

$$\int_a^b f(x)dx = \frac{3h}{8}\{f_0 + 3f_1 + 3f_2 + 3f_3 + \cdots + 3f_{n-1} + f_n\} + \frac{h^4(b-a)f^{(4)}(\zeta)}{80}$$

Remark: **(n) must be divided by 3**.

Example 6 ►

Use composite Simpson's rule to find the approximate value of the integration

$\int_0^1 x^2 e^x dx$. Use $n = 9$.

Solution $h = \frac{b-a}{n} = \frac{1-0}{9} = {}^{1}/_{9}$

n	x_n	$f(x_n) = x_n^2 e^{x_n}$	w_n	$w_n\, f(x_n)$
0	0	0	1	0
1	${}^{1}/_{9}$		3	
2	${}^{2}/_{9}$		3	
3	${}^{3}/_{9}$		2	
4	${}^{4}/_{9}$		3	
5	${}^{5}/_{9}$		3	
6	${}^{6}/_{9}$		2	
7	${}^{7}/_{9}$		3	
8	${}^{8}/_{9}$		3	
9	${}^{9}/_{9} = 1$		1	
$\sum$				

$$\int_0^1 x^2 e^x dx = \frac{3}{8} h \sum_{n=0}^{9} w_n\, f(x_n) = \frac{3}{8}\left(\frac{1}{9}\right)(\quad) =$$

Exercise

1- Use the composite Trapezoidal rule with the indicated value of n to approximate the following integrals and then estimate the maximum error

(1) $\int_1^3 (x^2+1)dx, \quad n=4$ (2) $\int_1^{1.6} (2x-1)dx, \quad n=6$

(3) $\int_2^3 \sqrt{x^3+1}dx, \quad n=4$ (4) $\int_0^{\pi} \sqrt{\sin x}\, dx, \quad n=6$

2- Use composite Simpson's rule to find the approximate value of the integration

$\int_1^2 \frac{1}{x} dx$. Use $n=10$.

3- Use both composite Trapezoidal and Simpson's rule to find the approximate value of the integration

$\int_{1.8}^{2.2} xe^x dx$. Use $n=4$.

4- Approximate $\int_0^2 x^2 e^{x^2} dx$ using h=0.25

1) Use the composite Trapezoidal rule.
2) Use the composite Simpson's rule.

5- Determine the values of n and h require to approximate $\int_0^2 e^{2x} \sin(3x)\, dx$ within error 10^{-4}

1) Use the composite Trapezoidal rule.
2) Use the composite Simpson's rule.

6- Use the composite Simpson's rule with the indicated value of n to approximate the following integrals and then estimate the maximum error

(1) $\int_0^1 \frac{1}{\sqrt{1+x^2}} dx, \quad n=4$ (2) $\int_0^{0.6} \frac{1}{\sqrt{4-x^2}} dx, \quad n=6$